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Some spectral inequalities for triangle-free regular graphs

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Abstract. We give three general bounds on the diameter, degree and order of triangle-free regular graphs with bounded second largest eigenvalue. Next, we consider bipartite regular graphs and present another four inequalities that bound the order of such graphs in terms of their degree and their second largest eigenvalue. We also prove some consequences and indicate graphs for which the corresponding bounds are attained.

1. Introduction

Let G be a simple graph on n vertices. The characteristic polynomial (of the adjacency matrix) of G will be denoted by P_G, while the corresponding roots,

$$\lambda_1\ (=\lambda_1(G)) \geq \lambda_2\ (=\lambda_2(G)) \geq \cdots \geq \lambda_n\ (=\lambda_n(G)),$$

are just the *eigenvalues* of G. The collection of eigenvalues (with repetition) is called the *spectrum* of G. The *diameter* $\mathrm{diam}(G)$ is the maximum distance between any two vertices of G. If G is regular then its degree will be denoted by r $(= r_G)$ (in this case we shall say that G is r-regular). A graph consisting of k disjoint copies of an arbitrary graph G will be denoted by kG, while the complement of G will be denoted by $\overline{G}$. A path and a cycle on n vertices will be denoted by P_n and C_n, respectively. A complete bipartite graph with parts of size n_1 and n_2 is denoted by K_{n_1,n_2}.

For the remaining terminology and notation we refer to [2] and [3].

In this paper we consider two classes of regular graphs: triangle-free, and (specially) bipartite regular graphs. We obtain a sequence of spectral inequalities on these graphs and give some of their consequences. Most of the bounds obtained are sharp, as is illustrated in the last section, where we list many graphs for which the corresponding bounds are attained.

The paper is organized as follows. In Section 2 we give some general bounds on the diameter, degree and order of triangle-free regular graphs with bounded second largest eigenvalue (Theorem 2.1 - Theorem 2.3), and we also give some simple implications (Corollary 2.1, Corollary 2.2, and Theorem 2.4). In Section 3 we consider bipartite regular graphs and present some inequalities that bound the order of such graphs in terms of its degree and its second largest eigenvalue (Theorem 3.1 - Theorem 3.3). Again, we obtain a several consequences (Corollary 3.1, and Theorem 3.4 - Theorem 3.7). Some comments are given in Section 4.

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2. Triangle-free regular graphs

First we prove a simple, yet useful, fact.

Theorem 2.1. *Let G be a connected triangle-free r-regular graph satisfying $\lambda_2 \leq \sqrt{r}$. Then* $\mathrm{diam}(G) \leq 4$*, where the equality can be attained only if $\lambda_2 = \sqrt{r}$.*

Proof Assume first that $\mathrm{diam}(G) = k \geq 6$, and let $v_1, \ldots, v_{k+1}$ be the diametral path. Both endvertices of the diametral path must have $r-1$ additional neighbours, and since G is triangle-free, $r-2$ of neighbours of v_1 can be eventually adjacent to v_3, and $r-2$ of neighbours of v_{k+1} can be eventually adjacent to v_{k-1}. Consider the subgraph, say H, of G determined by the vertices of the diametral path and the neighbours of its endvertices. By deleting $v_4, \ldots, v_{k-2}$ we obtain two graphs, both proper supergraphs of $K_{1,r}$, thus their indices being greater than $\sqrt{r}$. Using the Interlacing Theorem ([3], Corollary 1.3.12), we get $\lambda_2(H) > \sqrt{r}$, and therefore $\lambda_2(H) > \sqrt{r}$.

If $\lambda_2(G) < \sqrt{r}$ then $\mathrm{diam}(G) < 4$ (otherwise, G contains $2K_{1,r}$ as an induced subgraph causing $\lambda_2(G) \geq \sqrt{r}$).

Finally, let $\lambda_2(G) = \sqrt{r}$, and $\mathrm{diam}(G) = 5$. Consider the subgraph H of G determined by the vertices of the diametral path and the neighbours of its endvertices. Since H is the induced subgraph of G, $\lambda_2(H) \leq \sqrt{r}$, and since H contains $2K_{1,r}$ as an induced subgraph we have $\lambda_2(H) \geq \sqrt{r}$. Thus, in this case $\lambda_2(H) = \sqrt{r}$ must hold, which means that $P_H(\sqrt{r}) = 0$. Let $v_1, \ldots, v_6$ be the diametral path, let H_1 (resp. H_2) be the subgraph of H determined by v_1, v_2, v_3 and the $r-1$ neighbours of v_1 (resp. v_4, v_5, v_6 and the $r-1$ neighbours of v_6). Both graphs H_1 and H_2 are obtained by identifying the vertex of degree $r-k_i$ in $K_{1,r-k_i}$ with a vertex of degree k_i in K_{2,k_i}, where $1 \leq k_i \leq r-1$, $i = 1, 2$. Using [3, Theorem 2.2.3] we compute $P_{H_i}(\sqrt{r}) = -k_i^2\sqrt{r}^{r-2} < 0$, and then the value of the characteristic polynomial of H in the same point is computed by [3, Theorem 2.2.4]: $P_H(\sqrt{r}) = P_{H_1}(\sqrt{r})P_{H_2}(\sqrt{r}) > 0$, a contradiction. Thus, $\mathrm{diam}(G) \leq 4$, and the proof is complete. □

The next theorem will provide an upper bound on r, but first we prove a lemma. In both proofs we deal with the graph H_k obtained by attaching k pendant edges to each of endvertices of P_2. It is not difficult to see that $\lambda_2(H_k) = \frac{-1+\sqrt{1+4k}}{2}$.

Lemma 2.1. *Let G be a connected non-bipartite triangle-free r-regular graph satisfying $\lambda_2 \leq c$. If $r > 2c^2 + c + 1$, then G contains C_5 as an induced subgraph.*

Proof Assume to the contrary (i.e. C_5 is not an induced subgraph of G). Due to Theorem 2.1, we have $\mathrm{diam}(G) \leq 3$. Since G is not bipartite it must contain C_7 as an induced subgraph. Let $v_1, \ldots, v_7$ be the vertices of C_7 (given in natural order). Both vertices v_4 and v_5 can have at most $\lfloor c^2 \rfloor$ neighbours at distance 3 from v_1 (otherwise, G contains $2K_{1,\lfloor c^2 \rfloor+1}$), and so both v_4 and v_5 must have at least $\lceil c^2 + c \rceil + 1$ neighbours at distance 2 from v_1. There must be two adjacent vertices among the neighbours of v_4 and the neighbours of v_5 (at distance 2 from v_1), otherwise G contains $H_{\lceil c^2+c \rceil+1}$ causing $\lambda_2(G) > c$. These two adjacent vertices, their neighbours adjacent to v_1, and v_1 form C_5 in G. A contradiction! □

Theorem 2.2. *Let G be a connected non-bipartite triangle-free r-regular graph satisfying $\lambda_2 \leq c$ $(c > 0)$. Then, $r \leq c^4 + 2c^3 + 4c^2 + 2c + 3$.*

Proof Let $r > c^4 + 2c^3 + 4c^2 + 2c + 3$. Assume that there are two vertices of G at distance 2, u and v, that have at most $\lfloor c^2 + c \rfloor$ common neighbours. Then there are at least $\lceil c^4 + 2c^3 + 3c^2 + c + 3 \rceil$ neighbours of u not adjacent to v, and so there are more than $\lfloor c^2 + c \rfloor + 1$ groups of $\lceil c^2 \rceil$ neighbours of u not adjacent to v. Vertices in each of these groups must be adjacent to more than $\lceil c^4 + 2c^3 + 2c^2 + c + 1 \rceil$ neighbours of v (otherwise, G contains P_3 with $\lceil c^2 \rceil$ vertices attached to one of its endvertices, and $\lceil c^2 \rceil + 1$ to another causing $\lambda_2(G) > c$).

So in every group there is at least one vertex adjacent to at least $\lfloor c^2 + c \rfloor + 1$ neighbours of v. Fix exactly $\lfloor c^2 + c \rfloor + 1$ of such vertices, and let w denote one common neighbour of u and v. Any of fixed vertices can have at most $\lfloor c^2 + c \rfloor$ common neighbours with w (otherwise, G contains a graph $H_{\lfloor c^2+c \rfloor+1}$), but then there are more than $\lfloor c^2 + c \rfloor + 1$ neighbours of w which are not adjacent to any of those $\lfloor c^2 + c \rfloor + 1$ vertices. Thus, G contains $H_{\lfloor c^2+c \rfloor+1}$ as an induced subgraph, and consequently if $r > c^4 + 2c^3 + 4c^2 + 2c + 3$ holds, every two vertices at distance 2 must have at least $\lfloor c^2 + c \rfloor + 1$ common neighbours. By Lemma 2.1, G contains C_5 as an induced subgraph. Considering two adjacent vertices of C_5 and their neighbours we get that G must contain $H_{\lfloor c^2+c \rfloor+1}$ as an induced subgraph, which implies $\lambda_2(G) > c$, and the proof is complete. □

We can now formulate two simple consequences of Lemma 2.1 and Theorem 2.2, that provide spectral condition for bipartiteness in the class of triangle-free (triangle-free and pentagon-free) regular graphs. In both cases second largest eigenvalue is used to check whether a given triangle-free or triangle-free and pentagon-free regular graph is bipartite.

Corollary 2.1. *Let G be a connected triangle-free and pentagon-free r-regular graph satisfying $r > 2\lambda_2^2 + \lambda_2 + 1$. Then G is bipartite.*

Corollary 2.2. *Let G be a connected triangle-free r-regular graph satisfying $r > \lambda_2^4 + 2\lambda_2^3 + 4\lambda_2^2 + 2\lambda_2 + 3$. Then G is bipartite.*

An upper bound on n is given in the next theorem.

Theorem 2.3. *Given a triangle–free r–regular graph G on n vertices. Then*

$$n \leq \frac{r^2(\lambda_2 + 2) - r\lambda_2(\lambda_2 + 1) - \lambda_2^2}{r - \lambda_2^2}, \tag{1}$$

whenever the right hand side is positive.

Proof We partition the vertices of G into three parts: (i) an arbitrary vertex v, (ii) the vertices adjacent to v and (iii) the remaining vertices, and consider the corresponding blocking $A = (A_{ij})$, $1 \leq i, j \leq 3$ of its adjacency matrix. If d denotes the average vertex degree of the subgraph induced by the vertices non–adjacent to v, then the matrix whose entries are the average row sums in A_{ij} has the form

$$B = \begin{pmatrix} 0 & r & 0 \\ 1 & 0 & r-1 \\ 0 & r-d & d \end{pmatrix},$$

where B_{22} and B_{23} are computed using the condition that G is triangle–free, i.e. the subgraph induced by the vertices adjacent to v is totaly disconnected. By [5, Corollary 2.3], the eigenvalues of B interlace those of A. Since the characteristic polynomial of B is $(x - r)(x^2 - (d - r)x - d)$, we get that $x^2 - (d - r)x - d$ must be positive in λ_2, i.e. we have

$$\lambda_2^2 - (d - r)\lambda_2 - d \geq 0.$$

Or equivalently,

$$d \leq \frac{\lambda_2(\lambda_2 + r)}{\lambda_2 + 1}.$$

Since there are exactly r^2 edges between the set of vertices adjacent to v and the set of the remaining vertices of G, we get

$$d = \frac{r(n - 2r)}{n - r - 1}.$$

(Since G is triangle–free, we have $n - 2r \geq 0$.) Substituting this into the previous inequality we get

$$r(n - 2r)(\lambda_2 + 1) \leq (n - r - 1)\lambda_2(\lambda_2 + r). \tag{2}$$

After simplifying we obtain the result. □

Remark 2.1. *The inequality (2) holds for any graph described in the previous theorem, while there is a possibility that the right hand side of (1) is negative – the smallest example is a 3-regular graph of order 10 whose second largest eigenvalue is equal to 2 (it can be found in [3, Table A5] under the identification number 018).*

Finally, we can formulate the following direct consequence.

Theorem 2.4. *For every fixed λ_2, the set of all connected non-bipartite triangle-free regular graphs satisfying $\lambda_2 < \sqrt{r}$ is finite.*

Proof By Theorem 2.2, the degree r is bounded. Then, by Theorem 2.3, the order n is bounded, and the proof follows. □

3. Bipartite regular graphs

We shall need the following definition. The *bipartite complement* of connected bipartite graph G with two colour classes U and W is bipartite graph $\overline{\overline{G}}$ with the same colour classes having the edge between U and W exactly where G does not.

If G is a connected bipartite r-regular graph on $2n$ vertices, then $\overline{\overline{G}}$ is bipartite $(n-r)$-regular graph. Their adjacency matrices are

$$A(G) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}, \text{ and } A(\overline{\overline{G}}) = \begin{pmatrix} 0 & J-B \\ J-B^T & 0 \end{pmatrix},$$

respectively (J denotes all-1 matrix). It is easy to check that the characteristic polynomials of G and $\overline{\overline{G}}$ satisfy

$$\frac{P_G(x)}{x^2-r^2} = \frac{P_{\overline{\overline{G}}}(x)}{x^2-(n-r)^2}. \tag{3}$$

(compare [10, Theorem 4.1]). Considering the above equality we get that, apart from the eigenvalues $\pm r$ of G and $\pm(n-r)$ of $\overline{\overline{G}}$, the spectra of G and $\overline{\overline{G}}$ are the same. Note that if G is disconnected bipartite regular graph then its bipartite complement is not uniquely determined, but even then the above formula remains unchanged.

We give an upper bound on λ_2.

Theorem 3.1. *Let G be a connected bipartite r-regular graph on n vertices. Then $\lambda_2 \leq \frac{n}{2} - r$.*

Proof The bipartite complement of G is a bipartite $(\frac{n}{2}-r)$-regular graph, and thus its second largest eigenvalue can be at most $\frac{n}{2}-r$. Since the second largest eigenvalues of bipartite regular graph and its bipartite complement are the same, the proof follows. □

It is known from [4] that if G is a regular graph of order n and degree r, then the sum of its two largest eigenvalues r and λ_2 is at most $n-2$. Moreover, $r+\lambda_2 = n-2$ if and only if the complement of G has a connected component which is bipartite. A direct consequence of the previous theorem is a similar characterization if G is bipartite and r-regular.

Corollary 3.1. *Given a bipartite r-regular graph of order n, then the sum of its two largest eigenvalues r and λ_2 is at most $\frac{n}{2}$. Moreover, $r+\lambda_2 = \frac{n}{2}$ if and only if its bipartite complement is disconnected.*

The next theorem bounds the order of a bipartite r-regular graph whose diameter is equal to 3.

Theorem 3.2. *Let G be a connected bipartite r-regular graph on n vertices satisfying* $\mathrm{diam}(G) = 3$. *Then $n \leq 2\frac{r^2-\lambda_2^2(G)}{r-\lambda_2^2(G)}$, whenever the right hand side is positive.*

Proof Consider the distance partition (compare [5, p. 13]) of the set of vertices of G. It is an equitable partition with the quotient matrix

$$A = \begin{pmatrix} 0 & r & 0 & 0 \\ 1 & 0 & r-1 & 0 \\ 0 & \frac{2r(r-1)}{n-2} & 0 & \frac{r(n-2r)}{n-2} \\ 0 & 0 & r & 0 \end{pmatrix}.$$

By [5, Corollary 2.3], the eigenvalues of A interlace those of G. Thus, we have $\lambda_2(G) \geq \lambda_2(A) = \sqrt{\frac{r(n-2r)}{n-2}}$, which, after simplifying, leads to the result, and the proof is complete. □

The right hand side of the inequality above can be negative (an example is a graph from Remark 2.1). An immediate consequence of the previous theorem and Theorem 2.1 is that the order n of every bipartite r-regular graph whose second largest eigenvalue satisfies $\lambda_2 < \sqrt{r}$, satisfies $n \leq 2\frac{r^2-\lambda_2^2}{r-\lambda_2^2}$.

We now prove the following result.

Theorem 3.3. *Let G be a connected bipartite r-regular graph on n vertices whose second largest eigenvalue satisfies* $\lambda_2^4 < r$. *Then* $2(r+\lambda_2) \leq n \leq 2(r+\lambda_2^2)$. *If* $n = 2(r+\lambda_2^2)$, *then* $r \leq \lambda_2^2(\lambda_2^2-1)^2$.

Proof The left hand inequality is the consequence of Theorem 3.1. For the right hand inequality, we have $n \leq 2\frac{r^2-\lambda_2^2}{r-\lambda_2^2} = 2\Big(r+\lambda_2^2+\frac{\lambda_2^4-\lambda_2^2}{r-\lambda_2^2}\Big)$, and so $r > \lambda_2^4$ implies $n \leq 2(r+\lambda_2^2)$.

Suppose $n = 2(r+\lambda_2^2)$, and fix one vertex v of G. One colour class of G, say V, consists of r vertices adjacent to v, and the remaining λ_2^2 vertices which are at distance 3 from v, and the other colour class, say U, consists of v and the remaining $r+\lambda_2^2-1$ vertices. Denote by C the subset of the vertices of U that are adjacent to all of the vertices in V which are not adjacent to v. We have

$$r\lambda_2^2 \leq \lambda_2^2|C| + (\lambda_2^2-1)(r+\lambda_2^2-1-|C|),$$

implying $|C| \geq r-(\lambda_2^2-1)^2$.

Now consider $\overline{\overline{G}}$: it is a connected bipartite λ_2^2-regular graph whose second largest eigenvalue is also equal to λ_2, and due to Theorem 2.1, $\mathrm{diam}(\overline{\overline{G}}) \leq 4$. So, every vertex adjacent to v in G, can be adjacent to at most λ_2^2-1 vertices of $|C|$ in $\overline{\overline{G}}$. Thus

$$|C|(r-\lambda_2^2) \geq r\Big(|C|-(\lambda_2^2-1)\Big),$$

which implies $|C| \leq \frac{\lambda_2^2-1}{\lambda_2^2}r$.

Further, we get

$$r-(\lambda_2^2-1)^2 \leq |C| \leq \frac{\lambda_2^2-1}{\lambda_2^2}r,$$

which implies $r \leq \lambda_2^2(\lambda_2^2-1)^2$, and the proof follows. □

Remark 3.1. *We have seen that non-bipartite triangle-free regular graphs with bounded second largest eigenvalue have bounded degree (Theorem 2.2). This is not the case with bipartite regular graphs. Moreover, many examples can be constructed, say* $\lambda_2(K_{n,n}) = 0$ *and* $r_{K_{n,n}} = r$ *hold for any* $n \geq 2$. *In the next theorem we provide another family.*

Theorem 3.4. *For every* $k \in \mathbb{N}$, *and every* $r > k, r \in \mathbb{N}$, *there is a connected bipartite r-regular graph of diameter* 3 *whose second largest eigenvalue is equal to k.*

Proof Consider a disconnected graph whose one component is a complete bipartite graph $K_{k,k}$, and the other is any bipartite k-regular graph on $2r$ vertices (such a graph can always be obtained by removing of $r-k$ appropriate perfect matchings from $K_{r,r}$). It is now easy to see that its bipartite complement is bipartite r-regular graph, whose diameter is equal to 3, and whose second largest eigenvalue is equal to k. □

Note that the proof of the previous theorem also gives the construction of the described graphs. Using it, it is not difficult to construct many of them, in particular those with $\lambda_2 = k \leq 2\sqrt{r-1}$. Such graphs are often (good) expanders, i.e. sparse graphs with high connectivity properties [6]. Recall that a connected regular graphs whose second largest eigenvalue in modulus does not exceed $2\sqrt{r-1}$ are widely investigated Ramanujan graphs [3, Chapter 3.5.2].

In what remains, we consider the intervals for λ_2 that do not contain integers.

Lemma 3.1. *If G is a connected bipartite r-regular graph satisfying* $\mathrm{diam}(G) \geq 5$, *then* $\mathrm{diam}(\overline{\overline{G}}) = 3$.

Proof Since $\mathrm{diam}(G) \geq 5$ we have $n \geq 4r + 2$ [1], so every two vertices from the same colour class of G must have at least one common non-neighbour. This implies that every two vertices from the same colour class have at least one common neighbour in $\overline{\overline{G}}$. Consequently, $\overline{\overline{G}}$ is connected and $\mathrm{diam}(\overline{\overline{G}}) \leq 3$. Finally, if $\mathrm{diam}(\overline{\overline{G}}) < 3$ then G must be disconnected. □

Let further $\mathcal{R}$ denote the set of all connected bipartite regular graphs.

Theorem 3.5. *Let k be any non-integer greater than 1. The set $\mathcal{S} = \{G \in \mathcal{R} : \lambda_2 \in (\lfloor k \rfloor, k], r > \lambda_2^2\}$ is finite if and only if the set $\mathcal{T} = \{G \in \mathcal{R} : \lambda_2 \in (\lfloor k \rfloor, k], r \leq \lambda_2^2\}$ is finite.*

Proof Consider the following sets: $\mathcal{X} = \{G \in \mathcal{R} : \lambda_2 \in (\lfloor k \rfloor, k], \lambda_2^2 < r \leq \lambda_2^4\}$, $\mathcal{Y} = \{G \in \mathcal{R} : \lambda_2 \in (\lfloor k \rfloor, k], r > \lambda_2^4\}$, $\mathcal{Z} = \{G \in \mathcal{R} : \lambda_2 \in (\lfloor k \rfloor, k], r \leq \lambda_2^2\}$.

Graphs belonging to $\mathcal{X}$ have bounded degree: $\lfloor k \rfloor^2 < r \leq k^4$, their diameter is equal to 3 (by Theorem 2.1), and thus their order is bounded (by Theorem 3.2). Therefore, the set $\mathcal{X}$ is finite.

Graphs belonging to $\mathcal{Y}$ have bounded order (by Theorem 3.3): $n \leq 2(r + \lambda_2^2)$. Hence, the bipartite complement of any graph $G \in \mathcal{Y}$ has degree at most λ_2^2 which means that bipartite complements of graphs belonging to $\mathcal{Y}$ belong to $\mathcal{Z}$, and if $\mathcal{Z}$ is finite so is $\mathcal{Y}$. Thus, if $\mathcal{T}$ is finite then $\mathcal{S}$ is finite.

Due to Lemma 3.1, the set $\mathcal{Z}$ can be divided into four disjunct sets: $\mathcal{Z}_1 = \{G \in \mathcal{Z} : \mathrm{diam}(G) \leq 4\}$, $\mathcal{Z}_2 = \{G \in \mathcal{Z} : \mathrm{diam}(G) \geq 5, \overline{\overline{G}} \in \mathcal{X}\}$, $\mathcal{Z}_3 = \{G \in \mathcal{Z} : \mathrm{diam}(G) \geq 5, \overline{\overline{G}} \in \mathcal{Z}_1\}$, and $\mathcal{Z}_4 = \{G \in \mathcal{Z} : \mathrm{diam}(G) \geq 5, \overline{\overline{G}} \in \mathcal{Y}\}$. The set $\mathcal{Z}_1$ is finite because graphs in $\mathcal{Z}_1$ have degree bounded by k^2 and their diameter is 4 or 3. The sets $\mathcal{Z}_2$ and $\mathcal{Z}_3$ are finite, too (because they contain graphs whose bipartite complements belong to finite sets $\mathcal{X}$ or $\mathcal{Z}_1$). Thus, if $\mathcal{Y}$ is finite then $\mathcal{Z}_4$ is finite, and so is $\mathcal{Z}$. Consequently, if $\mathcal{S}$ is finite then $\mathcal{T}$ is finite. □

We get an immediate consequence of the previous theorem.

Theorem 3.6. *Let k be any non-integer greater than 1. The set $\{G \in \mathcal{R} : \lambda_2 \in (\lfloor k \rfloor, k]\}$ is finite whenever one of sets $\{G \in \mathcal{R} : \lambda_2 \in (\lfloor k \rfloor, k], r > \lambda_2^2\}$ or $\{G \in \mathcal{R} : \lambda_2 \in (\lfloor k \rfloor, k], r \leq \lambda_2^2\}$ is finite.*

For $k < 2$ we have the following result.

Theorem 3.7. *Let $k \in (1, 2)$. Then the set $\{G \in \mathcal{R} : 1 < \lambda_2 \leq k\}$ is finite.*

Proof Let $G \in \mathcal{R}$. If $\lambda_2(G) \leq k$ we get $\lambda_2(P_{\mathrm{diam}(G)+1}) \leq k$, causing $\mathrm{diam}(G) \leq \frac{2\pi}{\arccos \frac{k}{2}} - 2$. Now, the set $\{G \in \mathcal{R} : 1 < \lambda_2 \leq k, r \leq \lambda_2^2\}$ is finite since r is bounded by k^2, and the diameter of each belonging graph is bounded by $\left\lfloor \frac{2\pi}{\arccos \frac{k}{2}} - 2 \right\rfloor$. The application of Theorem 3.6 gives the result. □

4. Some comments

We list some graphs for which the bounds from the previous two sections are attained. Some of graphs listed are well known by their name, and they can be found in the literature.

Theorem 2.1 – This bound is attained for C_8, Möbius-Kantor graph, Pappus graph or Hadamard graphs (see [7]).

Theorem 2.2 – It seems that there is no graph attaining this bound. On the other hand, it gives a good estimation for r when λ_2 is small (say, $\lambda_2 = 1$; compare results of [8]). Its theoretical importance is pointed in Remark 3.1.

Theorem 2.3 – This bound is attained for the Petersen graph or the complement of the Clebsch graph (see [8] or [9]).

Theorem 3.1 – The equality holds for any complete bipartite regular graph on at least 4 vertices or $\overline{\overline{(r+1)K_2}}, r \geq 1$ (complete bipartite regular graphs with a perfect matching removed). It is also attained for $\overline{\overline{2G}}$, where G is any bipartite regular graph.

Theorem 3.2 – The equality holds for the Heawood graph and its bipartite complement. It is also attained for $\overline{\overline{(r+1)K_2}}, r \geq 1$ or graphs $H_4, \overline{\overline{H_4}}, H_5$, and $\overline{\overline{H_5}}$ listed in [8, Table 2].

Theorem 3.3 – Both equalities hold for $\overline{\overline{(r+1)K_2}}, r \geq 1$. The second equality also holds for the bipartite complement of the Pappus graph or the bipartite complement of the Möbius-Kantor graph.

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